The Mechanism of Callable Bull/Bear Contracts

Abstract

This paper studies the mechanism of Callable Bull/Bear Contracts (CBBCs). We show that the newly issued CBBCs are almost equivalent to leveraged positions on their underlying assets in a low interest rate environment. Their deltas are almost equal to 1 for bull contracts (-1 for bear contracts), while their vegas are almost equal to zero. The property that their prices are insensitive to the change of volatility explains why the products became very popular in Hong Kong after financial tsunami.

1 Introduction

The Callable Bull/Bear Contracts (CBBCs), also known as turbo warrants, are structured products that have some embedded exotic options, such as barrier, one-touch and lookback or Asian options. They were first listed in Hong Kong Exchange and Clearing Limited (HKEx) on 12 June 2006. The market size of the CBBCs has been growing rapidly since then. For example, the number of CBBCs increased more than 70 times from 24 in 2006 to 1,692 in 2009, which constituted one quarter of the total number of listed securities in HKEx. The turnover value of the CBBCs has even surpassed that of derivative warrants in 2009. In the HKEx securities market, the market share of the turnover of the CBBCs increased from 0.3% in 2007 to 10.9% in 2009, while that of derivative warrants decreased from 21.8% in 2007 to 10.7% in 2009, see Table 1 for details. The derivative warrants were the most popular derivatives in Hong Kong before the financial tsunami. Why did investors change their trading preference from derivative warrants to the CBBCs after the financial tsunami? In this paper, we will answer this question after studying the mechanism of the CBBCs.

There are some existing studies on turbo warrants. Eriksson (2005) derives explicit pricing formulas in the Black-Scholes (1973) framework for the turbo warrants that were introduced by Société Générale (SG) in 2005. Wong and Lau (2008) investigate the pricing of turbo warrants on currency exchange rates. Wong and Chan (2008) consider the pricing of the same type of turbo warrants studied by Eriksson (2005) under more advanced underlying models such as the CEV model, the fast mean-reverting stochastic volatility model of Fouque, Papanicolaou and Sircar (2000), and the two-scale volatility model of Fouque and Han (2003). They also examine the sensitivity of turbo warrants price to the implied volatility surface, and conclude that it is only true for the Black-Scholes model that the price of turbo warrant is not sensitive to the implied volatility. They demonstrate that the sensitivity of turbo warrant price to the implied volatility surface is actually model-dependent. With their results, we find it difficult to explain the popularity of the CBBCs after the financial tsunami.

In this paper, we find that the example used in Wong and Chan (2008) is not for the case of newly issued CBBCs in HKEx. We focus on studying the mechanism of newly issued CBBCs in order to answer the question raised above. We find that their prices are very close to the difference between the underlying price and the strike price in a low interest rate environment. Therefore, their deltas are almost equal to 1 for bull contracts (-1 for bear contracts), while their vegas are almost equal to zero. This means that the newly issued CBBCs could track the price changes of the underlying assets very closely, and also greatly reduce the influence of the implied volatility. The property that their prices are very insensitive to the implied volatility meets the need of investors, because they do not like volatility-sensitive products after the financial tsunami. This explains why CBBCs become so popular after the financial crisis.

The rest of the paper is organized as follows. Section 2 describes the structure of the CBBCs. Section 3 presents our theoretical and numerical results on the CBBCs. Section 4 concludes the paper.

2 The structure of CBBCs

2.1 CBBC product description

CBBCs are leveraged structured products written on an underlying asset (a stock or an index), and could closely track the performance of the underlying asset. Similar to regular European options, CBBCs have two types of contracts: bull or bear ones. For a Bull CBBC, the initial underlying price starts above a predetermined barrier level (or called "call price", "knock-out point" or "trigger point") and the contract behaves like a regular European call option if the underlying price never moves down to the barrier level before expiry. However, the contract will be called by the issuers if the barrier is touched before maturity, which is called Mandatory Call Event[2]. There are two categories of CBBCs: N and R. For a Category N CBBC, the barrier level is equal to the strike price, and the holders will not receive any payment once the Mandatory Call Event occurs before maturity. A Category R CBBC refers to the CBBC with the barrier level different from the strike price, and the holders will receive a small amount of residual value after a pre-specified short period when the Mandatory Call Event occurs at any time prior to expiry. The residual value is the positive amount of the settlement price less the strike price for a bull contract, or the positive amount of the strike price less the settlement price for a bear contract. The details of settlement method to calculate the residual value will be discussed in the next subsection.

In HKEx, the CBBCs are usually written on indices or a few single stocks. The CBBCs written on indices are the most popular ones. For example, CBBCs written on Hang Seng Index (HSI) constitute 55% of the total number of issued CBBCs, and CBBCs written on Hang Seng China Enterprises Index (HSCEI) constitute 12% of the total number, while the CBBCs written on single stocks constitute 29% of the total number. Most of the CBBCs traded in HKEx belong to Category R.

2.2 Products of similar nature

The turbo warrant with a barrier level at the strike price was first introduced in Germany in 2001. A more complex type of turbo warrants with a barrier level different from the strike price, paying a rebate (the same concept as CBBCs' residual value) when the barrier is touched, was introduced by SG in 2005. The former one is the same as Category N CBBCs, while the latter one is similar to Category R CBBCs. Turbo warrants are now popular in United Kingdom, Germany, Switzerland, Italy and Australia.

The first turbo warrant introduced in Germany is a knock-out barrier option with a barrier level at the strike price. For the turbo warrants with a residual value paid when

[2]The details are available at the following website: http://www.hkex.com.hk/eng/prod/secprod/cbbc/intro.htm

the barrier is hit, there are some differences for the settlement of the residual value among different financial institutions. For the turbo warrants introduced by SG in 2005, the settlement price of a turbo call (put) warrant is the minimum (maximum) of the underlying price during a three-hour period after the barrier is hit. For the turbo warrants issued by Citigroup in Australia, the settlement price for both turbo call and put is the arithmetic average of the underlying prices determined on each of the five trading days following the first-hit date. For the CBBCs traded in HKEx, the settlement price is determined in the listing document and must not be lower (higher) than the minimum (maximum) trading price of the underlying asset after the Mandatory Call Event and up to and including the next trading session for a bull (bear) contract. Usually, the issuers use the minimum (maximum) trading price of the underlying asset during the settlement period as the settlement price of a bull (bear) contract.

2.3 Model-free representation of the CBBCs

We now introduce two new concepts: Vanilla CBBCs and Exotic CBBCs. Vanilla CBBCs are referred to the CBBCs that pay a constant residual value calculated as the difference between the barrier level and the strike price when the barrier is hit before expiry. Exotic CBBCs are referred to the CBBCs that pay a complex residual value which could be viewed as a new contract, such as the three types of CBBCs mentioned in the subsection above.

Based on the Harrison and Pliska (1981) risk-neutral valuation formula, the price of a Vanilla Bull CBBC[3] at $t \leq T_b$ can be written as

$$BuVC_t = e^{-r(T-t)} E_t[\max\{S_T - K, 0\} 1_{\{T_b > T\}}] + E_t[e^{-r(T_b - t)}(S_b - K) 1_{\{T_b \leq T\}}], \qquad (1)$$

where S_t is the underlying price at time t, K is the strike price, S_b ($S_b \geq K$) is the barrier level, r is the risk-free rate, T is the maturity date, and $T_b = \inf\{t | S_t < S_b\}$ is the first time that S_t crosses the barrier level S_b. The current value of an Exotic Bull CBBC can

[3]When there is continuous dividend yield, the representations here and the results in the following sections require some modifications. However, the main results still hold.

be expressed as

$$BuEC_t = e^{-r(T-t)}E_t[\max\{S_T - K, 0\}1_{\{T_b>T\}}] + E_t[e^{-r(T_b+T_0-t)}\max\{F_{T_b}^{T_0} - K, 0\}1_{\{T_b\leq T\}}], \quad (2)$$

where T_0 is the length of the settlement period after the first hit, $F_{T_b}^{T_0}$ is the settlement price determined from the underlying price process during the settlement period from T_b to $T_b + T_0$.

For the turbo warrants introduced by SG in 2005 and the CBBCs traded in HKEx, $F_{T_b}^{T_0}$ is the minimum of the underlying price during the settlement period for a bull contract:

$$F_{T_b}^{T_0} = \min_{T_b\leq t\leq T_b+T_0} S_t. \quad (3)$$

For the turbo warrants issued by Citigroup in Australia, $F_{T_b}^{T_0}$ is the discretely sampled arithmetic average of the underlying prices during the settlement period after the barrier is hit:

$$F_{T_b}^{T_0} = \frac{1}{n}\sum_{i=1}^{n} S_{T_b+\frac{iT_0}{n}}, \quad (4)$$

where n is the total number of sampled spot prices during the settlement period.

The first part of equations (1) and (2) is a down-and-out call (DOC) option, the price of which is denoted as

$$DOC_t = e^{-r(T-t)}E_t[\max\{S_T - K, 0\}1_{\{T_b>T\}}]. \quad (5)$$

For Category N Bull Contracts, the strike price K is equal to the barrier level S_b and then the second part vanishes. In other words, a Category N Bull CBBC is simply a down-and-out call option.

It is easy to find that a Vanilla Bull CBBC could be replicated by a down-and-out call and a one-touch put. Since the Black-Scholes formulas for down-and-out and one-touch options are available[4], we are able to obtain the Black-Scholes price for a Vanilla Bull

[4] They are included in Appendix A for convenience.

CBBC as follows:

$$\begin{aligned} BuVC_t^{BS} &= DOC_t^{BS} + (S_b - K)OTP_t^{BS} \\ &= S - S\frac{K}{S_b}[N(-d_3) + bN(-d_6)] - Ke^{-r\tau}[N(d_4) - aN(-d_5)], \end{aligned} \quad (6)$$

where

$$a = \left(\frac{S_b}{S}\right)^{-1+\frac{2r}{\sigma^2}}, \quad b = \left(\frac{S_b}{S}\right)^{1+\frac{2r}{\sigma^2}}, \quad (7)$$

$$d_3 = \frac{\ln\frac{S}{S_b} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_4 = d_3 - \sigma\sqrt{\tau}, \quad (8)$$

$$d_5 = \frac{\ln\frac{S}{S_b} - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_6 = d_5 - \sigma\sqrt{\tau}, \quad (9)$$

$$\tau = T - t, \quad (10)$$

where OTP_t^{BS} stands for the Black-Scholes price of the one-touch put, and $N(\cdot)$ stands for the cumulative standard normal distribution function. The definitions of a, b, τ, $N(\cdot)$ and d_3 to d_6 will be used in the subsequent formulas without further explanations.

For the Exotic CBBC with settlement price $F_{T_b}^{T_0} = \min_{T_b \le t \le T_b + T_0} S_t$, its price can be expressed as (Eriksson 2005, Wong and Chan 2008)

$$BuEC_t^{BS} = DOC_t^{BS} + [LB_0^{BS}(S_b, K, T_0) - LB_0^{BS}(S_b, S_b, T_0)]OTP_t^{BS}, \quad (11)$$

where $LB_0^{BS}(S, m, T_0)$ stands for the Black-Scholes price at time 0 of a standard floating strike lookback call option with a realized minimum m and time to maturity T_0. The details of the pricing formula for the lookback call are also included in Appendix A for convenience.

3 Main results

3.1 The relationship between Vanilla CBBCs and Exotic CBBCs

Following Wong and Chan (2008), we look at the design of the residual value of the CBBCs. A simple and straightforward method is to use the barrier level as the settlement price and pay a constant rebate of $S_b - K$ when the barrier is hit. However, since the settlement cannot be finished immediately, some issuers choose to use the arithmetic average of the underlying prices during the settlement period as the settlement price, and some may only use the minimum (maximum) of the underlying price to settle the payment for a bull (bear) contract. Therefore, we should study the influence of different settlement methods on the pricing of CBBCs. In other words, we should analyze the price difference between Vanilla and Exotic CBBCs[5]. We define "residual ratio" as $(S_b - K)/(S_t - K)$ to measure the proportion of the residual part in the value of the whole contract.

Proposition 1 *The price difference between Vanilla and Exotic CBBCs mainly depends on the settlement of the residual part. If the underlying asset follows a continuous stochastic process, the difference vanishes when the settlement period or the residual ratio goes to zero.*[6]

The proposition can be proved easily by using the fact $S_b = S_{T_b}$ when the underlying asset follows a continuous stochastic process. If we change the settlement price to be the spot price that crosses the barrier level for the first time, the proposition will still hold for an underlying process with jumps.

With this proposition, we know that the price of a Vanilla CBBC is very close to that of an Exotic CBBC when the settlement period is very short and the residual value is a small part of the value of the whole contract. For the CBBCs traded in HKEx, the settlement period is less than one day, and the residual ratios for newly issued CBBCs are less than 20% since newly issued CBBCs are always deep-in-the-money. In Table 2, we provide two examples of Category R Bull CBBCs that were issued recently. For the CBBCs written

[5]Since it is difficult to find a closed-form solution for the Exotic CBBCs with discretely sampled arithmetic average of the underlying prices as the settlement price (for example, the turbo warrants issued by Citigroup in Australia), we will focus on the Exotic CBBCs that use the minimum (maximum) of the underlying price during the settlement period as the settlement price for a bull (bear) contract. For convenience, Exotic CBBCs in the following sections refer to this particular type of Exotic CBBCs.

[6]For the Exotic CBBCs with discretely sampled arithmetic average of the underlying prices as the settlement price, the conclusion can be changed to "the difference vanishes when the settlement period goes to zero".

on the Hang Seng Index, the residual value, $S_b - K$, is always 500 Hong Kong dollars, which is usually one-tenth of the value of the whole newly issued CBBCs. For the CBBCs written on single stocks, the residual value is also a very small portion of the whole contract. Therefore, the price of a Vanilla type should be very close to that of an Exotic type for newly issued CBBCs in HKEx.

In Table 3, we present a numerical comparison between the example in Wong and Chan (2008) and a newly issued CBBC in HKEx . The settlement period of the example in Wong and Chan (2008) is 0.2 year, nearly two and a half months, and its residual ratio is 50%. For the newly issued CBBC in HKEx, the settlement period is no more than one day and the residual ratio is about 10%. Due to the significant differences on the two aspects, the percentage differences between Vanilla and Exotic types are 35.11% for the example in Wong and Chan (2008) and only 1.73% for the CBBC traded in HKEx.

In order to demonstrate the sensitivity of Vanilla CBBCs and Exotic CBBCs to the settlement period and the residual ratio, we make further numerical analysis, which is shown in Table 4. Since Hong Kong interbank offer rate (HIBOR) for 9 months is about 0.5% in 2010, we use 0.5% as the risk-free rate in our computation. The volatility level here is 30%, which is close to the historical volatility before the CBBC's issue date. We change the settlement period from one day to one month, and also change the barrier level to alter the residual ratio from 10% to 90%. The result shows that the prices of Vanilla CBBCs and Exotic CBBCs are indeed very close when the settlement period is very short and the residual part is a relative small portion of the whole contract.

Therefore, for the newly issued CBBCs in HKEx, we can use the price of a Vanilla one as a good approximation of that of an Exotic one.[7] In the following sections, we will focus on the price of Vanilla ones in order to capture the main features of the CBBCs.

[7] If the spot price goes down to a level close to the barrier, the relative difference between the Vanilla and Exotic CBBCs seems to be large, but the absolute difference is always a small portion of the overall initial value as the newly issued CBBCs in HKEx are deep-in-the-money. Therefore using a Vanilla CBBC to represent an Exotic CBBCs does not make much difference in the overall profit-and-loss analysis.

3.2 Vanilla CBBCs at zero risk-free rate

For Vanilla CBBCs, we have the following model-free result at zero risk-free rate.

Proposition 2 *If the underlying asset follows a continuous stochastic process and $r = 0$, then the price of a Vanilla Bull CBBC at $t \leq T_b$ is equal to $S_t - K$.*

Proof. When $r = 0$, equation (1) can be simplified as

$$BuVC_t = E_t[(S_T - K)1_{\{T_b > T\}}] + E_t[(S_b - K)1_{\{T_b \leq T\}}]. \tag{12}$$

Rewriting the formula above gives

$$BuVC_t = E_t[S_T 1_{\{T_b > T\}} + S_b 1_{\{T_b \leq T\}}] - K. \tag{13}$$

Since the underlying asset follows a continuous stochastic process, then we have $S_b = S_{T_b}$ and

$$BuVC_t = E_t[S_T 1_{\{T_b > T\}} + S_{T_b} 1_{\{T_b \leq T\}}] - K = S_t - K. \tag{14}$$

Q.E.D.

This proposition only requires that the underlying process is continuous, and holds for any stochastic volatility model without jump. It is easy to prove that if we change the definition of the settlement price as the spot price that crosses the barrier level for the first time, the proposition will hold for an underlying process with jumps.

The proposition shows that in a zero risk-free rate environment, the price of a Vanilla Bull CBBC has a simple expression $S_t - K$, i.e., a leveraged position on its underlying asset. In this case, Vanilla CBBCs have zero sensitivity with respect to the implied volatility since their Vegas are equal to zero. Furthermore, this proposition also shows that the risk-free rate should be a key factor in the pricing of Vanilla CBBCs. In the following subsection, we further study the impact of the risk-free rate on the pricing of Vanilla CBBCs by an asymptotic method.

3.3 Vanilla CBBCs at low risk-free rate

3.3.1 Asymptotic analysis under Black-Scholes model

It is not easy to observe the effect of the risk-free rate on the price of Vanilla CBBCs directly from formula (6), we make the sensitivity of Vanilla CBBCs to the risk-free rate explicit by using Taylor series expansion. In the following propositions, we provide approximate formulas under Black-Scholes model for the price, delta and vega of a Vanilla Bull CBBC in a low risk-free rate environment. Similar results for a Vanilla Bear CBBC can be obtained by the same method.

Proposition 3 *The second-order Taylor series expansion of a Vanilla Bull CBBC's price under Black-Scholes model around $r = 0$ is given by*

$$\begin{aligned} BuVC_t^{BS} = {} & S - K + \left[\left(S\frac{K}{S_b}F_1 - KF_2\right) + \tau K\right] r \\ & - \left[\frac{1}{2}\left(\frac{2}{\sigma^2}\ln\frac{S}{S_b} + \tau\right)\left(S\frac{K}{S_b}F_1 - KF_2\right) + \frac{\tau^2}{2}K\right] r^2 + O(r^3), \end{aligned} \tag{15}$$

where

$$F_1(S, S_b, \sigma, \tau) = \frac{2\sqrt{\tau}}{\sigma}N'(d) - \left(\frac{2}{\sigma^2}\ln\frac{S}{S_b} + \tau\right)N(-d), \tag{16}$$

$$F_2(S, S_b, \sigma, \tau) = \frac{2\sqrt{\tau}}{\sigma}N'(d') - \left(\frac{2}{\sigma^2}\ln\frac{S}{S_b} - \tau\right)N(-d'), \tag{17}$$

$$d = \frac{\ln\frac{S}{S_b} + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad d' = d - \sigma\sqrt{\tau}, \tag{18}$$

$$\tau = T - t. \tag{19}$$

Proposition 4 *The delta of a Vanilla Bull CBBC under Black-Scholes model is given by*

$$Delta_t^{BS} = 1 + \frac{K}{S_b}\left[\frac{2r}{\sigma^2}bN(-d_6) - \frac{S}{S_b}e^{-r\tau}b\left(\frac{2r}{\sigma^2} - 1\right)N(-d_5) - N(-d_3)\right]. \tag{20}$$

The second-order Taylor series expansion of a Vanilla Bull CBBC's delta under Black-Scholes model around $r = 0$ is

$$\begin{aligned} Delta_t^{BS} = {} & 1 + \frac{K}{S_b}\left[F_1 - \frac{2}{\sigma^2}\left(N(-d) - \frac{S_b}{S}N(-d')\right)\right] r \\ & - \frac{K}{S_b}\left[\frac{2}{\sigma^2}G + \frac{1}{2}\left(\frac{2}{\sigma^2}\ln\frac{S}{S_b} + \tau\right)F_1\right] r^2 + O(r^3), \end{aligned} \tag{21}$$

where

$$\begin{aligned} G(S, S_b, \sigma, \tau) =& \frac{\sqrt{\tau}}{\sigma}\left(N'(d) - \frac{S_b}{S}N'(d')\right) \\ &- \frac{2}{\sigma^2}\ln\frac{S}{S_b}\left(N(-d) - \frac{S_b}{S}N(-d')\right) - \tau N(-d), \end{aligned} \tag{22}$$

F_1, d, d' and τ are defined as before.

Proposition 5 *The vega of a Vanilla Bull CBBC under Black-Scholes model is given by*

$$Vega_t^{BS} \quad = \quad S\frac{K}{S_b}\frac{4b}{\sigma^3}\ln\frac{S}{S_b}\left[\frac{S}{S_b}e^{-r\tau}N(-d_5) - N(-d_6)\right]r. \tag{23}$$

The second-order Taylor series expansion of a Vanilla Bull CBBC's Vega under Black-Scholes model around $r = 0$ is

$$Vega_t^{BS} = S\frac{K}{S_b}\frac{4}{\sigma^3}\ln\frac{S}{S_b}\left[\left(N(-d) - \frac{S_b}{S}N(-d')\right)r + Gr^2\right] + O(r^3), \tag{24}$$

where G, d, d' and τ are defined as before.

Table 5 presents the accuracy of the zeroth, first and second order approximations of the price, delta and vega for a Vanilla Bull CBBC. For a relative small risk-free rate 0.5%, the zeroth approximation is very close to the exact value, and even for a relative large risk-free rate 4.5%, the errors from the approximations are also acceptable. Meanwhile, the first order and second order approximations are almost the same as the exact value under different risk-free rates. Therefore, formula (15) is a good approximation of the value of Vanilla Bull CBBCs under a low interest rate environment.

3.3.2 Asymptotic analysis under stochastic volatility framework

Proposition 2 holds for any models with a continuous underlying process. Therefore, we could examine the sensitivity of Vanilla CBBCs to the risk-free rate under stochastic volatility framework.

Following Heston (1993), we assume that the spot asset at time t follows the diffusion process:

$$dS = rSdt + \sqrt{v}Sdz_1, \tag{25}$$

where z_1 is a Wiener process and the risk-free rate r is constant. Assume that variance v, the square of volatility, follows a square-root process (Cox, Ingersoll, and Ross 1985):

$$dv = \kappa(\theta - v)dt + \sigma_v \sqrt{v} dz_2, \tag{26}$$

where z_2 has correlation ρ with z_1.

No-arbitrage principle dictates that the price of a Vanilla Bull CBBC, $C(S, v, t)$, must satisfy the following partial differential equation (PDE) with boundary and final conditions:

$$\left\{ \begin{aligned} &\frac{\partial C}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma_v v S \frac{\partial^2 C}{\partial S \partial v} + \frac{1}{2} \sigma_v^2 v \frac{\partial^2 C}{\partial v^2} + r S \frac{\partial C}{\partial S} \\ &+ [\kappa(\theta - v) - \lambda v] \frac{\partial C}{\partial v} - rC = 0, \ S_b < S < +\infty, \ 0 < t < T, \\ &C(S, v, T) = S - K, \ S_b < S < +\infty, \\ &C(S_b, v, t) = S_b - K, \ 0 < t < T, \end{aligned} \right. \tag{27}$$

where λ is the market price of volatility risk.

Defining the linear differential operators L_1 and L_2 as follows

$$L_1 = \frac{\partial}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2}{\partial S^2} + \rho \sigma_v v S \frac{\partial^2}{\partial S \partial v} + \frac{1}{2} \sigma_v^2 v \frac{\partial^2}{\partial v^2} + +[\kappa(\theta - v) - \lambda v] \frac{\partial}{\partial v}, \tag{28}$$

$$L_2 = 1 - S \frac{\partial}{\partial S}, \tag{29}$$

then the PDE in (27) becomes $L_1 C = r L_2 C$.

Using perturbation method, we are able to obtain an expression for the price of a Vanilla CBBC. Let $C = C_0 + rC_1$, and since $C_0 = S - K$ satisfies the PDE with $r = 0$, boundary and final conditions in (27), then we obtain the following equation for C_1

$$L_1 C_1 = L_2 C_0, \tag{30}$$

which is equivalent to the following PDE:

$$L_1 C_1 + K = 0. \tag{31}$$

Therefore, $C_1(S, v, t)$, the first-order part of the asymptotic expansion of a Vanilla Bull

CBBC's price, must satisfy the following PDE with boundary and final conditions:

$$\begin{cases} \dfrac{\partial C_1}{\partial t} + \dfrac{1}{2} v S^2 \dfrac{\partial^2 C_1}{\partial S^2} + \rho \sigma_v v S \dfrac{\partial^2 C_1}{\partial S \partial v} + \dfrac{1}{2} \sigma_v^2 v \dfrac{\partial^2 C_1}{\partial v^2} \\ +[\kappa(\theta - v) - \lambda v] \dfrac{\partial C_1}{\partial v} + K = 0, \ S_b < S < +\infty, \ 0 < t < T, \\ C_1(S, v, T) = 0, \ S_b < S < +\infty, \\ C_1(S_b, v, t) = 0, \ 0 < t < T. \end{cases} \tag{32}$$

The first-order approximation of a Vanilla Bull CBBC's price can be expressed as

$$C(S, v, t) = S - K + C_1(S, v, t) r + O(r^2). \tag{33}$$

Without deriving a detailed formula for $C_1(S, v, t)$, we are able to conclude that as long as the risk-free rate is small, the price of a Vanilla Bull CBBC will be very close to $S_t - K$ even under stochastic volatility models. Wong and Chan (2008) assert that the implied volatility is insignificant to turbo warrant pricing is only true for the Black-Scholes model and show that the sensitivity of turbo warrant to the implied volatility surface is actually model-dependent. However, for the newly issued CBBCs in HKEx under a low risk-free rate environment, the price of a Vanilla CBBC is very close to that of an Exotic one. Therefore, we could assert that the prices of newly issued CBBCs are indeed very insensitive to the implied volatility when the risk-free rate is relative small. The reason why Wong and Chan (2008) find an opposite result is that the example used in their paper has a long settlement period and a large residual ratio, which is not the case for newly issued CBBCs in HKEx.

3.4 Reasons for the popularity of CBBCs after financial tsunmai

Based on the results in the previous subsection, we can make the following statement for the newly issued Bull CBBCs in HKEx under a low interest rate environment: "Their prices are very close to $S_t - K$, which implies their deltas are almost equal to one and their vegas are almost equal to zero." The results for bear contracts are similar. Therefore, the price changes of newly issued CBBCs tend to follow very closely to the price changes of the underlying assets, which can be observed clearly in Figure 1. Moreover, their prices are very insensitive to the implied volatility of the underlying assets, which can be seen clearly in Figure 2. This also explains why the CBBCs in HKEx are typically issued at a price that represents the difference between the spot price of the underlying asset and the strike price, plus a small amount of funding cost.

In Table 6, we present a numerical comparison between European Calls, Vanilla CBBCs and Exotic CBBCs under different risk-free rates and volatilities in order to show the sensitivity of the three types of options with respect to the risk-free rate and volatility. It can be seen clearly that when the volatility level increases from 10% to 90%, the price of an European call option increases rapidly while the prices of Vanilla CBBCs and Exotic CBBCs slightly decrease. Moreover, when the volatility level is within a normal range, e.g., below 50%, the price of a Vanilla CBBC is very close to that of an Exotic CBBC.

With our study on the mechanism of the CBBCs, we are now able to provide a reasonable explanation for the popularity of CBBCs in Hong Kong after the financial crisis. During the financial tsunami, the risk-free rate was very low and the market volatility rose more sharply than any time before. Under such an environment, the newly issued CBBCs represent leveraged positions on the underlying asset. It could closely track the price changes of the underlying assets, and also greatly reduce the influence of the implied volatility. The property that their prices are very insensitive to the implied volatility meets the need of investors that they do not like volatility-sensitive products, especially after the financial tsunami. This explains why CBBCs became so popular after the financial crisis.

4 Conclusion

In this paper, we study the mechanism of Callable Bull/Bear Contracts (CBBCs). We find that the difference between Vanilla CBBCs and Exotic CBBCs mainly depends on the settlement of the residual part. When the settlement period is short and the residual part is a relative small portion of the whole contract, which is the case for the newly issued CBBCs in HKEx, the prices of Vanilla CBBCs and Exotic CBBCs are very close. Therefore, we could focus on analyzing Vanilla CBBCs to capture the effect of the risk-free rate on the sensitivity of CBBCs to the implied volatility.

If the risk-free rate is zero, we have an important model-free result for the Vanilla CBBCs: the price of a Vanilla Bull CBBC is equal to $S_t - K$, which means that its delta is equal to one and its vega is equal to zero. Furthermore, we show that the price of a Vanilla Bull CBBC is very close to $S_t - K$ by an asymptotic method if the risk-free rate is low. We conclude that Vanilla CBBCs are very insensitive to the volatility of underlying assets in a low interest rate environment. This property of CBBCs is very different from that of regular European options, and could explain why CBBCs grew so rapidly during the past several years and became so popular after the financial crisis. During the financial tsunami, the risk-free rate was low and the volatility level was very high. Under such an environment, the CBBCs are much more insensitive to the implied volatility than derivative warrants, regular European options. This property meets the need of investors that they did not like volatility-sensitive products during the financial tsunami. That is why the investors in Hong Kong preferred trading the CBBCs rather than trading derivative warrants after the financial tsunami.

Appendix

A Black-Scholes formulas for the prices of down-and-out options, standard floating strike lookback options and one-touch options

1. For a down-and-out call option, if $K \leq S_b$,

$$DOC_t^{BS} = S[N(d_3) - bN(-d_6)] - Ke^{-r\tau}[N(d_4) - aN(-d_5)], \tag{34}$$

else if $K > S_b$,

$$DOC_t^{BS} = S[N(d_1) - bN(-d_8)] - Ke^{-r\tau}[N(d_2) - aN(-d_7)], \tag{35}$$

where

$$a = \left(\frac{S_b}{S}\right)^{-1+\frac{2r}{\sigma^2}}, \quad b = \left(\frac{S_b}{S}\right)^{1+\frac{2r}{\sigma^2}}, \tag{36}$$

$$d_1 = \frac{\ln\frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}, \tag{37}$$

$$d_3 = \frac{\ln\frac{S}{S_b} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_4 = d_3 - \sigma\sqrt{\tau}, \tag{38}$$

$$d_5 = \frac{\ln\frac{S}{S_b} - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_6 = d_5 - \sigma\sqrt{\tau}, \tag{39}$$

$$d_7 = \frac{\ln\frac{SK}{S_b^2} - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_8 = d_7 - \sigma\sqrt{\tau}, \tag{40}$$

$$\tau = T - t, \tag{41}$$

and the function $N(\cdot)$ is the cumulative standard normal distribution function.

2. For a standard floating strike lookback call option,

$$\begin{aligned} LB_0^{BS}(S, m, T_0) = {} & SN(d_m) - me^{-rT_0}N(d_m - \sigma\sqrt{T_0}) \\ & - Se^{-rT_0}\frac{\sigma^2}{2r}\left[\left(\frac{S}{m}\right)^{-\frac{2r}{\sigma^2}} N(-d_m + \frac{2r}{\sigma}\sqrt{T_0}) - e^{rT_0}N(-d_m)\right], \end{aligned}$$

where

$$d_m = \frac{\ln \frac{S}{m} + (r + \frac{1}{2}\sigma^2)T_0}{\sigma\sqrt{T_0}}, \tag{42}$$

and m is the realized minimum of the underlying price.

3. For an one-touch put option ($S_0 > S_b$),

$$OTP_t^{BS} = \frac{S}{S_b}[N(-d_3) + bN(-d_6)], \tag{43}$$

where b, d_3 and d_6 are defined as before.

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Table 1: **Market information for Derivative Warrants and CBBCs from 2006 to 2009**

This table presents the product numbers, market values, turnover values and market shares of Derivative Warrants and the CBBCs traded in Hong Kong Exchange and Clearing Limited (HKEx) from 2006 to 2009. The data is from HKEx Fact Book.

Year	Derivative Warrants				CBBCs			
	Prod. Num.	Market Value*	Turnover Value*	Market Share (%)	Prod. Num.	Market Value*	Turnover Value*	Market Share (%)
2006	1,959	456,072	1,790,059	21.5	24	1,782	11,335	0.1
2007	4,483	696,996	4,693,860	21.8	131	20,406	71,380	0.3
2008	3,011	169,574	3,433,736	19.5	1,314	195,229	1,039,557	5.9
2009	3,367	136,441	1,654,895	10.7	1,692	91,191	1,676,065	10.9

*The unit for Market Values and Turnover Values is million HK dollars.

Table 2: **Two examples of the CBBCs traded in HKEx**

This table presents the main information of two Category R Bull CBBCs issued recently in HKEx. The data is collected from HKEx website.

Underlying:	Hang Seng Index (HSI)	China Life (02628)
Issuer:	HSBC	UBS
CBBC Code:	60129	62337
CBBC Category:	R Bull	R Bull
Issue Price (Initial Issue):	HKD 0.25	HKD 0.55
Strike Level:	15800	27.88
Barrier Level:	16300	28.88
Launch Date (Initial Issue):	25-01-2010	27-05-2010
Listing Date (Initial Issue):	01-02-2010	03-06-2010
Maturity Date:	29-09-2010	13-12-2010
Entitlement Ratio:*	20000	10
Spot Price**	20726.18	32.90
Market Price***	20000×0.25=HKD 5000	10×0.55=HKD 5.50
Residual ratio****	10.15%	19.92%

* The entitlement ratio represents the number of CBBCs required to be exercised into one share or one unit of the underlying asset.

** The spot price is the close price of the trading day prior to the launch day.

*** Market price is the market value of CBBCs to be exercised into one unit of the underlying asset.

**** Residual ratio, defined as $(S_b - K)/(S - K)$, measures the weight of residual value in the whole contract.

Table 3: **Comparison between the example in Wong and Chan (2008) and a newly issued CBBC in HKEx**

This table presents a numerical comparison between the example in Wong and Chan (2008) and a newly issued CBBC in HKEx. The prices are calculated under Black-Scholes model. In fact, Wong and Chan (2008) used $r = 5\%$ and $q = 5\%$ in their computation. r and q usually show up in the formula together as the term $(r - q)$. Then $(r - q)$ in Wong and Chan (2008) is zero, and $(r - q)$ here is also nearly zero since q is zero here. Therefore, the simplified parameters here will not change the main results.

Option	Example in Wong and Chan (2008)	CBBC in HKEx
Type	Call	Call
Spot price*	10	20,726
Strike price	8	15,800
Barrier level	9	16,300
Maturity T	1.00	0.677
Settlement period T_0	0.20	0.004
Risk-free rate	0.50%	0.50%
Volatility	30.00%	30.00%
European Call Price	2.38	5275.98
Intrinsic Value**	2.04	4979.55
Residual Ratio***	50.00%	10.15%
DOC Part	1.26	4787.20
Residual Part	0.76	182.89
Vanilla CBBC (I)	2.02	4970.09
DOC Part	1.26	4787.20
Residual Part	0.23	98.14
Exotic CBBC (II)	1.49	4885.35
(I-II)/II	35.11%	1.73%

* The spot price is the close price of the trading day prior to the launch day.

** The Intrinsic Value is defined as $\max(S - Ke^{-r\tau}, 0)$ for call.

*** Residual ratio, defined as $(S_b - K)/(S - K)$, measures the weight of residual value in the whole contract.

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Table 4: **Comparison between Vanilla CBBCs and Exotic CBBCs under different settlement periods and residual ratios**

We use a newly issued CBBC (Code 60129) in our computation: S=20726.18, K=15800, T=0.677, r=0.5% and σ=30%. The prices are calculated under Black-Scholes model. We change T_0 and S_b to study the sensitivity of the difference between Vanilla CBBCs and Exotic CBBCs with respect to the settlement period and the residual ratio. T_0=0.004, 0.02, and 0.083 are one day, one week and one month respectively. The last column is the percentage difference between Vanilla CBBCs and Exotic CBBCs.

	Barrier Level	Residual Ratio*	Vanilla (I)	Exotic (II)	(I-II)/II
T_0=0.004					
	16,300	10.15%	4970.09	4885.35	1.73%
	17,300	30.45%	4964.20	4834.39	2.69%
	18,300	50.75%	4956.14	4778.78	3.71%
	19,300	71.05%	4945.65	4715.46	4.88%
	20,300	91.35%	4932.56	4646.34	6.16%
T_0=0.020					
	16,300	10.15%	4970.09	4837.63	2.74%
	17,300	30.45%	4964.20	4680.20	6.07%
	18,300	50.75%	4956.14	4562.76	8.62%
	19,300	71.05%	4945.65	4434.98	11.51%
	20,300	91.35%	4932.56	4297.54	14.78%
T_0=0.083					
	16,300	10.15%	4970.09	4812.01	3.29%
	17,300	30.45%	4964.20	4489.34	10.58%
	18,300	50.75%	4956.14	4196.27	18.11%
	19,300	71.05%	4945.65	3923.10	26.06%
	20,300	91.35%	4932.56	3652.08	35.06%

* Residual ratio, defined as $(S_b - K)/(S - K)$, measures the weight of residual value in the whole contract.

Table 5: **Price, Delta and Vega of Vanilla CBBCs under different order approximations**

We use a newly issued CBBC (Code 60129) in our computation: S=20726.18, K=15800, S_b=16300, T=0.677, T_0=0.004 and σ=30%. The prices, deltas and vegas are calculated under Black-Scholes model.

r	0th Order (I)	1st Order (II)	2nd Order (III)	True Value (IV)	$\frac{(I-IV)}{IV}$	$\frac{(II-IV)}{IV}$	$\frac{(III-IV)}{IV}$
			Price				
0.50%	4926.18	4970.04	4970.09	4970.09	-0.88%	0.00%	0.00%
2.50%	4926.18	5145.47	5146.82	5146.74	-4.29%	-0.02%	0.00%
4.50%	4926.18	5320.91	5325.28	5324.82	-7.49%	-0.07%	0.01%
			Delta				
0.50%	1.0000	1.0039	1.0039	1.0039	-0.38%	0.00%	0.00%
2.50%	1.0000	1.0195	1.0186	1.0186	-1.83%	0.08%	0.00%
4.50%	1.0000	1.0350	1.0324	1.0323	-3.13%	0.26%	0.00%
			Vega*				
0.50%	0.000%	-0.014%	-0.014%	-0.014%	—	0.26%	0.01%
2.50%	0.000%	-0.066%	-0.065%	-0.065%	—	1.44%	0.16%
4.50%	0.000%	-0.115%	-0.113%	-0.112%	—	2.85%	0.52%

*The Vega, calculated as $\partial(\ln BuVC_t)/(100\partial\sigma)$ here, measures the percentage change of the price when volatility increases by 1%.

Table 6: **Comparison between European Calls, Vanilla CBBCs and Exotic CBBCs under different risk-free rates and volatilities**

We use a newly issued CBBC (Code 60129) in our computation: S=20726.18, K=15800, S_b=16300, T=0.677 and T_0=0.004. The prices are calculated under Black-Scholes model. We change r and σ to study the sensitivity of European Calls, Vanilla CBBCs and Exotic CBBCs with respect to the risk-free rate and the volatility.

	Volatility	Euro.C (I)	Vanilla (II)	Exotic (III)	(I-III)/III	(II-III)/III
r=0.5%						
	10.0%	4979.71	4979.52	4979.24	0.01%	0.01%
	30.0%	5275.98	4970.09	4885.35	8.00%	1.73%
	50.0%	6086.80	4958.29	4759.03	27.90%	4.19%
	70.0%	7067.57	4950.42	4673.86	51.21%	5.92%
	90.0%	8095.38	4945.16	4617.59	75.32%	7.09%
r=2.5%						
	10.0%	5191.32	5191.14	5190.97	0.01%	0.00%
	30.0%	5454.95	5146.74	5066.95	7.66%	1.57%
	50.0%	6231.58	5087.67	4892.92	27.36%	3.98%
	70.0%	7190.27	5048.00	4774.81	50.59%	5.72%
	90.0%	8202.07	5021.46	4696.44	74.64%	6.92%
r=4.5%						
	10.0%	5400.11	5399.98	5399.88	0.00%	0.00%
	30.0%	5634.17	5324.82	5249.81	7.32%	1.43%
	50.0%	6376.84	5218.53	5028.26	26.82%	3.78%
	70.0%	7313.24	5146.53	4876.72	49.96%	5.53%
	90.0%	8308.85	5098.36	4775.89	73.98%	6.75%

Figure 1: The values of a European Call, Intrinsic Value and a Vanilla CBBC as functions of underlying spot price. The information of the CBBC (Code 60129) is as follows: S=20726.18, K=15800, S_b=16300, T=0.677, T_0=0.004, r=0.5% and σ=30%. The prices are calculated under the Black-Scholes model. The Intrinsic Value is defined as $\max(S - Ke^{-r\tau}, 0)$ for call.

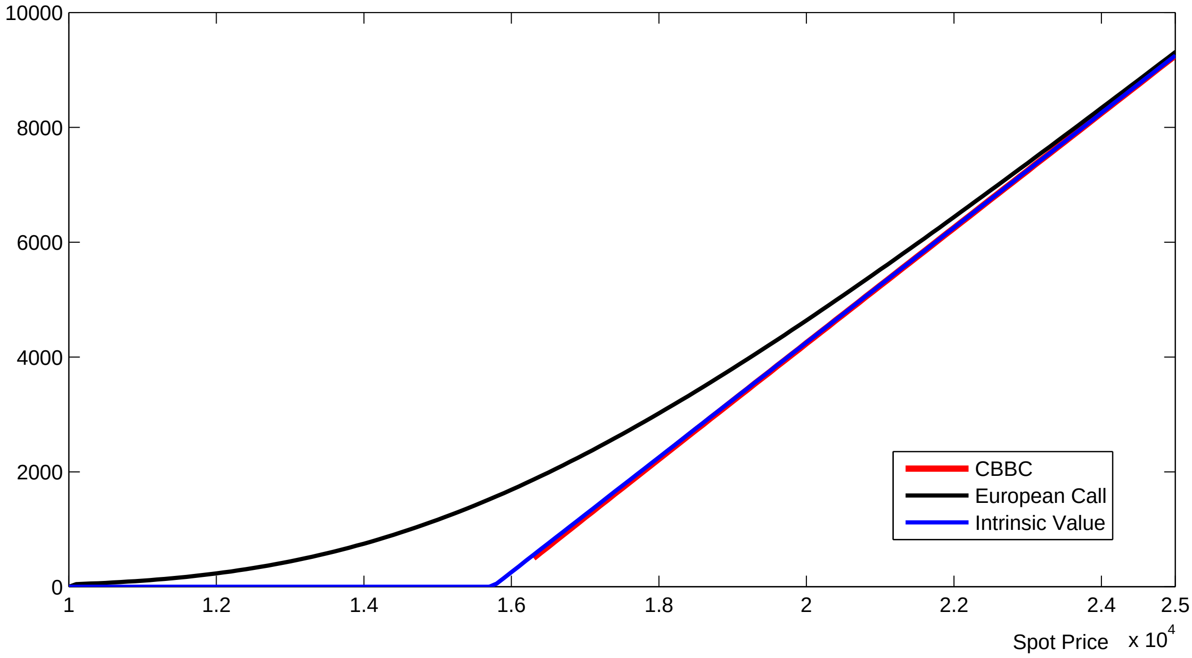

Figure 2: The price of a European call and a Vanilla CBBCs as functions of volatility. The information of the CBBC (Code 60129) is as follows: S=20726.18, K=15800, S_b=16300, T=0.677, T_0=0.004, r=0.5% and σ=30%. The prices are calculated under the Black-Scholes model.